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Numerical investigation of the interaction of coaxial vortex rings

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Introduction

For more than 100 years the problem of the interaction of vortex structures has been the subject of a large number of investigations. The area of application of the results obtained by the solution of this problem has changed from the creation of the vortex atom model[1] and forming of the vortex sheets[2] to the description of mixing processes, modern methods of weather forecasting and the theory of turbulence. Although some applications later proved intractable (for example the vortex atom model), the results from such attempts had the classical character as a rule and were employed in other domains of science. The possibility of visualization of the vortex phenomena led to corrections to the theoretic models at each stage of their development. The development of computer capabilities enabled a study of a variety of non-integrable cases of vortex interactions. They closely connect with the conception of order and chaos in the theory of dynamical systems. In vortex dynamics, a sensitivity to chaos leads to critical analysis of the possibilities of vortex methods being used for the description of different natural phenomena.

In this paper, the problem of interaction of the very simple vortex structures, namely the coaxial vortex rings, is analysed. The investigation of such non-linear interactions is restricted by the equations used and the accuracy of corresponding experiments. As the first step in the investigation of the interaction of several coaxial vortex rings, the inviscid Euler equations can be used. Their solution can exhibit chaotic behaviour. Moreover, their non-

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dissipative nature allows the use of many results obtained for Hamiltonian systems.

The process of interaction of two vortex rings had already been qualitatively described by Helmholtz[3]. Different situations of these interactions were examined in detail in the works of Dyson[4] and Hicks[5] at the end of the last and at the beginning of this century. At the same time, the first experimental results of Rogers[6], Oberbeck[7] and others were published.

A current great interest in vortex dynamics is again apparent. Different aspects of interaction of coaxial vortex rings were experimentally examined in the papers of Oshima *et al.*[8], Maxworthy[9], Yamada and Matsui[10], Auerbach[11] and Claus[12]. Theoretical and numerical investigations have been published in papers by Moore and Saffman[13,14], Widnall[15], Möhring[16], Kambe and Minota[17], Kambe[18], Müller and Obermeier[19], Shariff and Leonard[20], and also in the monographs of Saffman[21] and Meleshko and Konstantinov[22] and in the report of Shariff *et al.*[23].

Formulation of the problem

Cylindrical co-ordinates (r, ϑ, z) with radial distance r, azimuthal angle ϑ and axial distance z are introduced. Axisymmetry requires that $\partial/\partial\vartheta = 0$. The fluid is assumed to be incompressible, and there exists a stream function ψ such that

$$u_z = \frac{1}{r}\frac{\partial\psi}{\partial r}, \quad u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}.$$

In such a case, the vorticity has only one component $\omega_\vartheta = \omega$. For axisymmetric flow without swirl the equation for the azimuthal vorticity is

$$\frac{D(\omega/r)}{Dt} = 0.$$

In terms of the stream function, the azimuthal vorticity is

$$\omega = -\frac{1}{r}\frac{\partial^2\psi}{\partial z^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r^2}. \tag{1}$$

Each ring has initial co-ordinates $R_i^{(0)}$, $Z_i^{(0)}$, radius of circular core $a_i^{(0)} << R_i^{(0)}$ and some constant circulation $\kappa_i = \iint \omega drdz$. It is necessary to determine the position of each ring at any moment of time.

The stream function ψ for the point $Q(R_i - a_i \cos \alpha, \vartheta, Z_i + a_i \sin \alpha)$, which is taken on the surface of the ring i, can be written according to the Biot-Savart law[21,24]:

$$\psi(Q,t)=\frac{\kappa_i R_i}{2\pi}\left[\left(\log\frac{8R_i}{a_i}-2\right)-\left(\log\frac{8R_i}{a_i}-\frac{1}{4}\right)\frac{a_i}{2R_i}\cos\alpha+\ldots\right]+$$
$$+\sum_{j=1}^{N}\frac{\kappa_j}{2\pi}\left(I_{ij}+a_i\sin\alpha\frac{\partial I_{ij}}{\partial Z_i}-a_i\cos\alpha\frac{\partial I_{ij}}{\partial R_i}\right),$$

$$I_{ij}=\int_0^{\pi}\frac{R_iR_j\cos\vartheta' d\vartheta'}{\sqrt{(Z_i-Z_j)^2+R_i^2-2R_iR_j\cos\vartheta'+R_j^2}}\equiv\sqrt{R_iR_j}\,C(k_i),$$
$$C(k_{ij})=\left(\frac{2}{k_{ij}}-k_{ij}\right)K(k_{ij})-\frac{2}{k_{ij}}E(k_{ij}),$$
$$k_{ij}^2=\frac{4R_iR_j}{(Z_i-Z_j)^2+(R_i+R_j)^2}. \tag{2}$$

Here $K(k)$ and $E(k)$ denote the complete elliptic integrals of the first and second kind of modulus k.

It is assumed that the core radius $a_i(t)$ is variable. However, the core shape of the rings remains circular. With such an assumption, the equations of motion of N vortex rings can be written in the so-called Dyson's model form[4]:

$$\dot{Z}_i=\frac{\kappa_i}{4\pi R_i}\left(\log\frac{8R_i}{a_i}-\frac{1}{4}\right)+\frac{1}{\kappa_iR_i}\frac{\partial U}{\partial R_i},$$
$$\dot{R}_i=-\frac{1}{\kappa_iP}\frac{\partial U}{\partial Z_i}, \tag{3}$$

where

$$U=\frac{1}{2\pi}\sum_{i=1}^{N}{}'\kappa_i\kappa_jI_{ij}, \qquad a_i^2R_i=\text{constant},$$

and the prime indicates that summation is over all $i\neq j$. The problem was made dimensionless by referring all the variables to a length scale $[L]=R_0$, and time scale $[T]=2\pi R_0^2/\kappa_0$, characteristic of the physical problem.

This system of equations assumes that no deformation of core shapes occurs. A deformation of the core shape of isolated ring was investigated in the papers of Fraenkel[25] and Norbury[26]. An examination of the coaxial interaction of inviscid core deformation was made by Shariff *et al.*[23]. The relatively simple Dyson's model which can be considered as correct when $a_i << R_i$ is investigated here. A comparison with the experiments of leapfrog motion[10] with Dyson's model was made in the paper by Gurzhi *et al.*[27]. Deviations from the simpler model occurred only for later stages of the motion.

The system of equations (3) has two first integrals:

$$\sum_{i=1}^{N} \kappa_i R_i^2 \equiv P = \text{constant}$$

$$\sum_{i=1}^{N} \frac{\kappa_i^2 R_i}{4\pi}\left(\log \frac{8R_i}{a_i} - \frac{7}{4}\right) + U \equiv T = \text{constant} \tag{4}$$

Equation (3) is an example of a Hamiltonian system with canonical variables $p_i = \kappa_i R_i^2$ and $q_i = Z_i$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \tag{5}$$

with the independent Hamiltonian H

$$H = \sum_{i=1}^{N} \frac{\kappa_i^{3/2} p_i^{1/2}}{2\pi}\left(\log \frac{8\kappa_i^{-3/4} p_i^{3/4}}{a_i^{(0)} R_i^{(0)1/2}} - \frac{7}{4}\right) + \frac{1}{\pi} \sum_{i,j=1}^{N}{}' (\kappa_i \kappa_j)^{3/4} (p_i p_j)^{1/4} C(k_{ij})$$

$$k_{ij}^2 = \frac{4(\kappa_i \kappa_j p_i p_j)^{1/2}}{\kappa_i \kappa_j (q_i - q_j)^2 + (\kappa_j^{1/2} p_i^{1/2} + \kappa_i^{1/2} p_j^{1/2})^2}.$$

In addition to H the system (5) has another first integral $\sum_{i=1}^{N} p_i$ = constant. Therefore, this system is integrable for $N = 2$ according to Liouville's theorem. It is non-integrable for $N \geq 3$, because it has no additional first integrals.

The system of coaxial vortex rings has the centre of vorticity with co-ordinates

$$R_c = \left(\frac{\sum_{i=1}^{N} \kappa_i R_i^2}{\sum_{i=1}^{N} \kappa_i}\right)^{1/2}, \quad Z_c = \frac{\sum_{i=1}^{N} \kappa_i R_i^2 Z_i}{\sum_{i=1}^{N} \kappa_i R_i^2}. \tag{6}$$

The value $R_c(t) = R_c(0)$ = const for arbitrary initial conditions in the problem examined.

Interaction of two vortex rings

The problem of the interaction of two coaxial vortex rings had been investigated extensively. The motion of two vortex rings can be used as the test for different numerical models. If two vortex rings have the same sense of rotation, they travel in the same direction and ,under certain conditions, the rear vortex will attempt to pass through the front one.

When the vortex rings have the opposite sense of rotation, the so-called head-on collision takes place.

In the paper by Gurzhi *et al.*[27], the analysis of possibility of the leapfrogging of two rings had been made by using invariants (4). Such an approach permits us to obtain the exact analytic solution for the problem under consideration.

The motion of two coaxial vortex rings can be defined with the help of only three independent determined parameters $\chi = \kappa_1/\kappa_2$, $\rho_0 = R_i^{(0)}/R_0$ and $Z_0 = (Z_1^{(0)} - Z_2^{(0)})/R_0$. As R_0 the initial radius of ring $R_2^{(0)}$ was chosen. The conditions, when the leapfrogging of two rings takes place, can be determined by the Dyson's theorem[4]. In accordance with this theorem, the leapfrogging always takes place if the rings at the initial time have equal velocities and the distance between them is infinity. If Z_0 was given, the ρ_0 corresponding to leapfrogging is found from the conditions:

$$\frac{\chi}{R_1^{(\infty)}}\left(\frac{3}{2}\log\frac{R_1^{(\infty)}}{\rho_0} + \log\frac{8R_1^{(0)}}{a_1^{(0)}} - \frac{1}{4}\right) = \frac{1}{R_2^{(\infty)}}\left(\frac{3}{2}\log R_2^{(\infty)} + \log\frac{8R_2^{(0)}}{a_2^{(0)}} - \frac{1}{4}\right),$$

$$\chi R_1^{(\infty)2} + R_2^{(\infty)2} = \chi\rho_0 + 1,$$

$$\chi^2 R_1^{(\infty)}\left(\frac{3}{2}\log\frac{R_1^{(\infty)}}{\rho_0} + \log\frac{8R_1^{(0)}}{a_1^{(0)}} - \frac{7}{4}\right) + R_2^{(\infty)}\left(\frac{3}{2}\log R_2^{(\infty)} + \log\frac{8R_2^{(0)}}{a_2^{(0)}} - \frac{7}{4}\right) \le$$

$$\le \chi^2\rho_0 \log\frac{8R_1^{(0)}}{a_1^{(0)}} + \log\frac{8R_2^{(0)}}{a_2^{(0)}} + 2\chi\rho_0^{1/2} C(k_0). \tag{7}$$

In the last expression, the inequality means that the initial kinetic energy of the system cannot be less than the kinetic energy of two isolated rings because $C(k_0) > 0$. It is easy to show that independently of the value of Z_0 the relations (7) are satisfied when

$$\rho_0 = \chi\frac{\log\frac{8R_1^{(0)}}{a_1^{(0)}} - \frac{1}{4}}{\log\frac{8R_2^{(0)}}{a_2^{(0)}} - \frac{1}{4}}.$$

The domains of values Z_0 and ρ_0 for $a_i^{(0)}/R_i^{(0)} = 0.01$, where the leapfrogging takes place, are shaded in Figure 1(a)[27]. Here the domains for $\chi = 0.5$, 1.0, and 2.0 are marked by 1-3 respectively.

The case when $\chi < 0$ (so-called head-on collision) was studied in the paper of Gurzhi and Konstantinov[28]. It was shown that in the dependence of combinations of ρ_0, χ and Z_0 there are three cases of head-on collisions: direct scattering, limiting case of two symmetric rings and mutual trapping. All these cases are shown in Figure 1(b). The co-ordinates are the parameters $-\chi$ and ρ_1^*. The parameter ρ_1^* denotes the maximum value of the dimensionless radius of the first ring R_1 at the moment of collision (when $Z_1 - Z_2 = 0$). To each curve

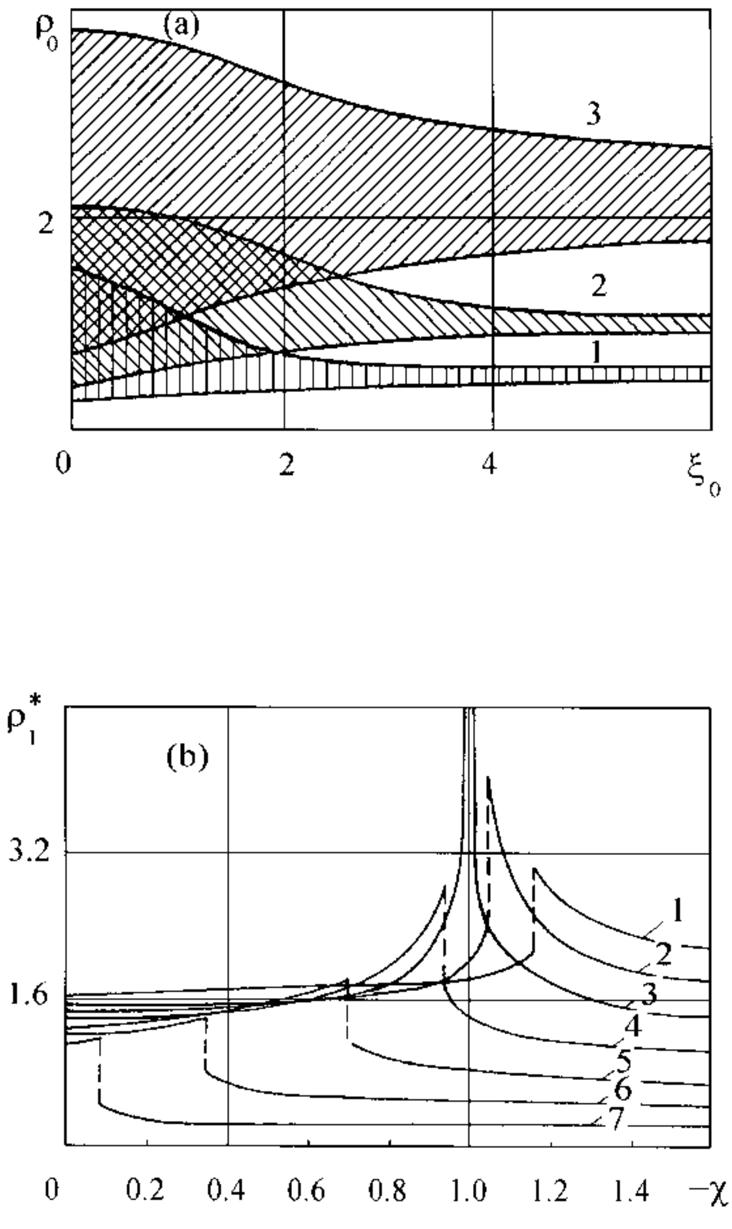


Figure 1.
Leapfrogging conditions (a) and classification of head-on collisions (b) by the interaction of two vortex rings

there corresponds a specific value of the initial ratio ρ_0 (In Figure 1(b) the numbers 1-7 denote the curves for ρ_0 = 1.4, 1.2, 1.0, 0.8, 0.6, 0.4 and 0.2). The initial distance Z_0 = 50 was chosen.

The case of direct scattering means that the both rings keep their initial identity after collision. In Figure 1(b), the solid parts of curves 1, 2 and 4-7 correspond to this case. The intervals of the curves $\rho_0^* = f(\chi)$ to the left of the dashed parts of lines correspond to the collisions when the ring R_2 passes through the ring R_1. The intervals to the right of the dashed parts of lines correspond to the collision when the ring R_1 passes through the ring R_2.

Limiting cases of symmetrical rings by $\chi = -1$ is well known and has been already described by Helmholtz[3]. As the rings approach each other, their radii increase without bound, and the distance decreases between them continuously. The rings never pass through each other.

The case of mutual trapping is of great interest and has been described in [28]. In this case, both rings move in the same direction in spite of the fact that they have opposite senses of rotation. The initial data corresponding to this case can be defined at the dashed parts of curves in Figure 1(b). To determine the initial conditions for such a case of motion is difficult. It is suitable to use $Z_0 = 0$ for that case. Such curious phenomena have been found by analytically solving system (4). In the case of two rings, this system allows us to observe the behaviour of radii R_i in depending on the distance between the rings $Z_{12} = Z_1 - Z_2$. It was found that two combinations of the initial meanings of ρ_0 and Z_0 for equal χ can make the same pairs of the first integrals P and T. In such situations, the motion cannot be determined definitely by first integrals. The first combination Z_{12} changes in limits $-Z_{12}^* < Z_{12} < Z_{12}^*$, and the other one in limits $-\infty < Z_{12} < +\infty$. The initial data of the first combination correspond to the dashed parts of curves $\rho_1^* = f(\chi)$ and determine mutual trapping of vortex rings. Data of the second combination correspond to direct scattering.

The examples for all possible situations of interaction of two vortex rings are presented in Figure 2.

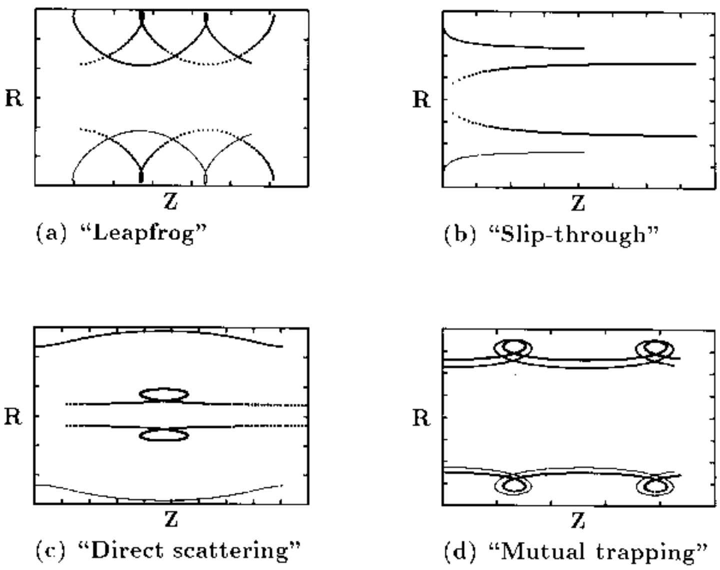


Figure 2.
All possible situations of interaction of two vortex rings

Numerical investigation of interaction of *N* vortex rings

System (3), with the associated initial conditions, was integrated numerically for different N values by the method of extrapolation with variable step and order[29]. The accuracy of the computed trajectories has been thoroughly investigated by integrating the system of equations with various initial step sizes and local error values. The initial step $\Delta t = 0.0005$ with maximum step $\Delta t_{max} = 0.001$ and a local error $\Delta R = 10^{-13}$ were sufficiently demeaned to ensure proper accuracy. Double precision computations were found necessary.

For the case $N \geq 3$, the number of independent determinative parameters increases considerably. For example, when $N = 3$ for the determination of all possible situations it would be necessary to use at least six independent parameters (if the parameters of all rings are divided by the parameters of one ring). At the same time, the system of the rings becomes very sensitive to changes in the initial conditions. In Figure 3(a), the simplest example of such sensitiveness is shown. Here the trajectories of motion for the simplest case of initial conditions $R_1^{(0)} = R_2^{(0)} = R_3^{(0)} = 1$, $Z_1^{(0)} = 0$, $Z_2^{(0)} = 1$, $Z_3^{(0)} = 2$ for equal κ are shown. The situation when rings 1 and 2 have identical initial co-ordinates, and ring 3 has initial co-ordinate $Z_3^{(0)} = 2.02$ is presented in Figure 3(b). As one can see, it is impossible to predict the process of the interaction even with such a little change of one of the co-ordinates. Only the trajectories of the first rings for both cases are shown in Figure 3(c). The trajectory corresponding to the case of Figure 3(a) by the solid line, and the trajectory corresponding to the case Figure 3(b) by the dashed one are marked.

The restricted class of the initial conditions of the vortex rings has been investigated in detail in[30]. The initial set of vortex rings has been determined by the formulae

$$R_i^{(0)} = R_0 - \rho_0 \cos\frac{(2i-1)\pi}{N}, \quad Z_i^{(0)} = \rho_0 \sin\frac{(2i-1)\pi}{N},$$
$$\kappa_i = \kappa_0, \quad i = 1, \ldots, N. \tag{8}$$

At the initial moment, central lines of ring cores were uniformly distributed on the surface of the ring with radius $R_0 = 1$ and radius ρ_0 of cross-section (Figure 4). The value of ρ_0 was the control parameter of the problem.

A variety of tools were applied to characterize the dynamics behaviour of the vortex trajectories. These include spectra, Poincaré sections and relative phase trajectories.

The behaviour of the spectral function was calculated for the co-ordinates $R_i(t)$ of each vortex ring. When the system of vortex rings in the process of interaction decayed into several subsystems, the spectral function was calculated for the velocity of the centre of vorticity $V_{Zc}(t)$. The spectrum was estimated by an FFT algorithm. The effective sampling period t was equal to 0.0016. To obtain a good spectral resolution, the FFT length was chosen to be

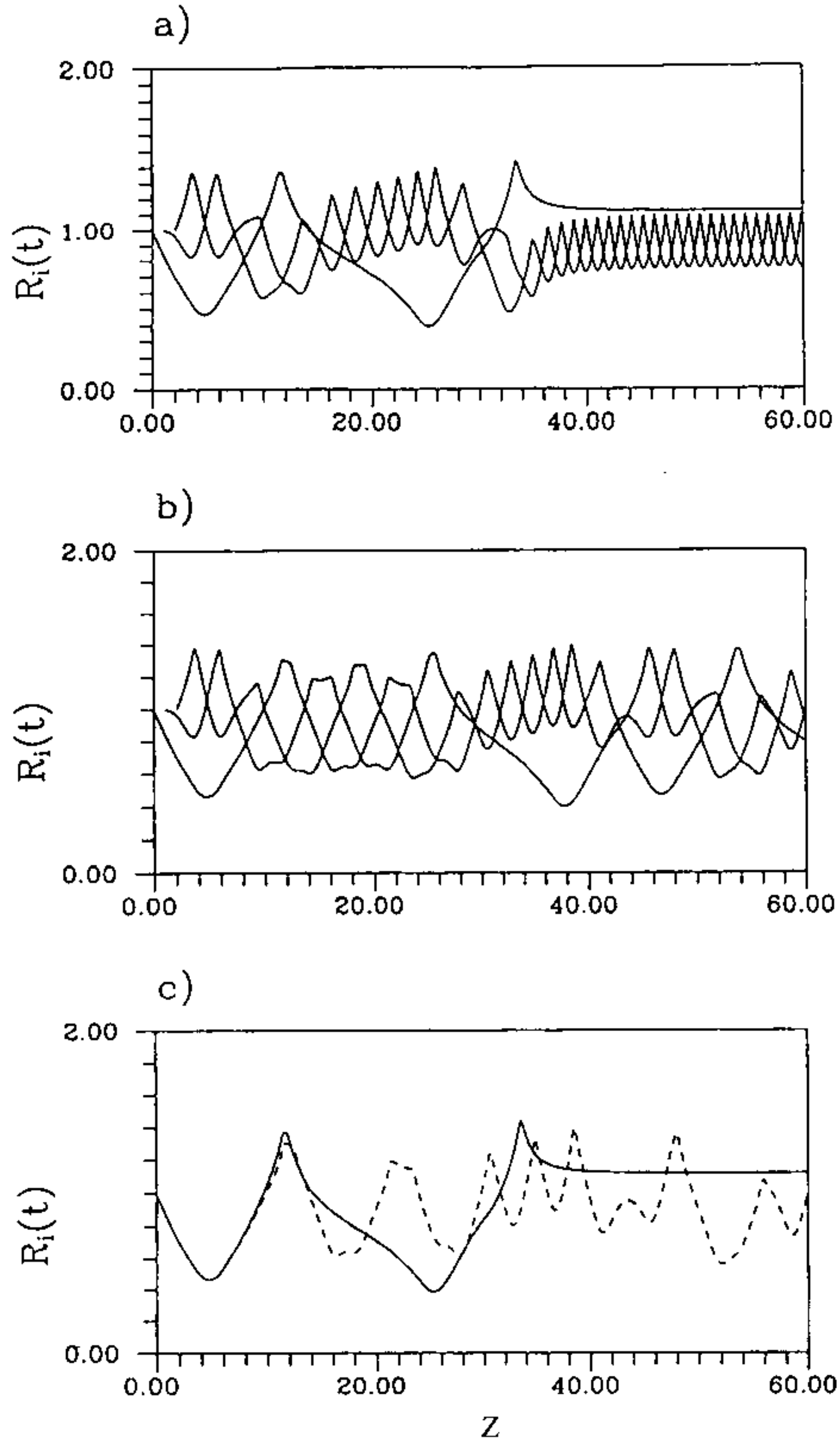


Figure 3.
Example of interaction of three vortex rings: (a) $R_i^{(0)} = 1$, $Z_1^{(0)} = 0$, $Z_2^{(0)} = 1$, $Z_3^{(0)} = 2$, $\kappa_i = 1$; (b) $R_i^{(0)} = 1$, $Z_1^{(0)} = 0$, $Z_2^{(0)} = 1$, $Z_3^{(0)} = 2.02$, $\kappa_i = 1$; (c) difference between trajectories $R_1 = f_1(Z)$ from case (a) (solid line) and $R_1 = f_2(Z)$ from case (b) (dashed line)

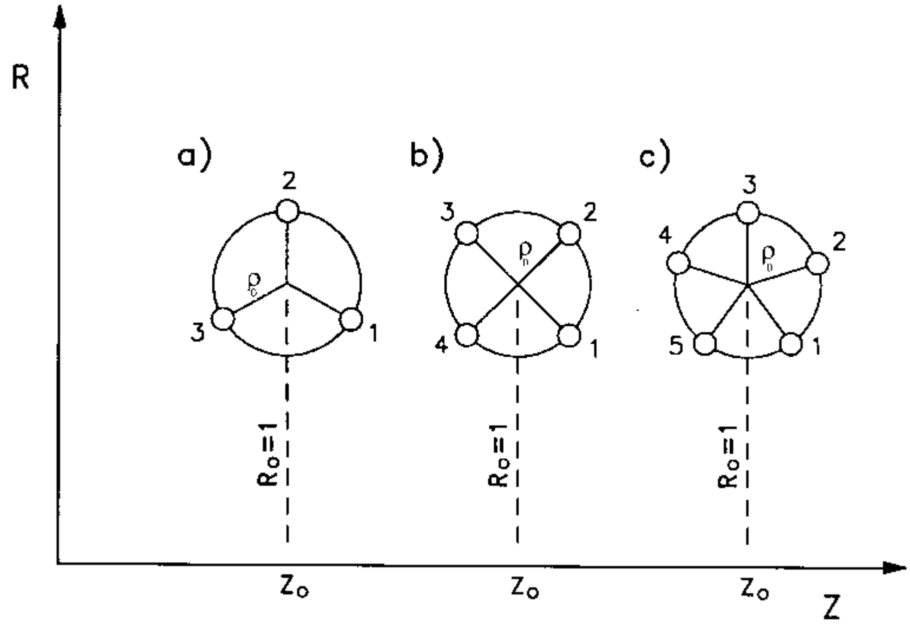


Figure 4.
Initial conditions of vortex rings for the cases (a) N = 3, (b) N = 4 and (c) N = 5

8,192 points for calculating the co-ordinates $R_i(t)$ and 16,384 points for calculating the velocity of the centre of vorticity $V_{Zc}(t)$.

For the cases under consideration, the system of rings moves in the same direction and its centre is at infinity. For generation of phase co-ordinates, the method of delays was chosen[31]. This method leads to the modified 1D Poincaré map by stroboscoping either the vortex position or the velocity of the vorticity centre at the intersection of the phase trajectory with a chosen surface. In this technique Poincaré sections were defined with the help of the following set of co-ordinates

$$R_i = R_i(t_i),$$
$$R_i' = R_i(t_i - \Delta t),$$
$$R_i'' = R_i(t_i - 2\Delta t) \tag{9}$$

The value of Δt was chosen equal to the period of the first mode of oscillations of the vortex ring. As a control surface the cylindrical surface R = 1 was chosen. If Poincaré sections were determined according to $V_{Zc}(t)$, the control surface was determined as the mean velocity of the vorticity centre. A sketch of the generating stroboscopic Poincaré sections is shown in Figure 5.

Relative phase trajectories have been determined in the space (Z_{iC}, R_i), where $Z_{iC} = Z_i - Z_c$.

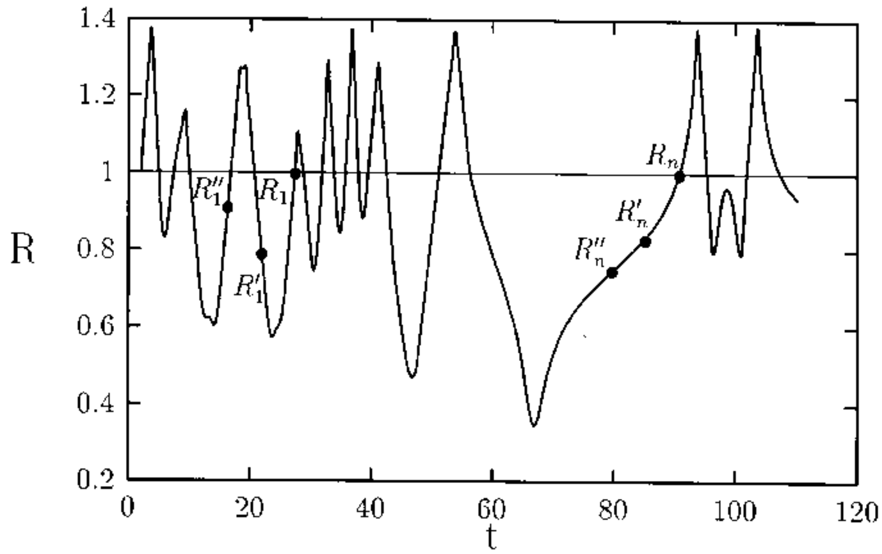


Figure 5.
Sketch for determining the Poincaré sections by the method of delay

The numerical calculations of the interaction of three rings ($a_i^{(0)}/R_i^{(0)}$ = 0.01) allow us to obtain some interesting results. There are three regions of value ρ_0 with regular motion, and each of them has different properties. In the centre of one region by ρ_0 = 0.23, the periodic motion of rings occurs (See Poincaré sections in Figure 6). Here all the sections (≈ 2,000 sections were calculated) are strictly situated on 13 points in the (R_2', R_2'') space. As shown in Figure 7, a perfect picture for ρ_0 = 0.23 takes place also for the relative phase trajectories. Between the regions with regular motion there is always one with chaotic characteristics. Beginning from the value $\rho_0 \geq 0.35$ the initial system, which consists of three vortex rings, decays after some interactions into two subsystems with a leapfrogging ring pair and one isolated ring (3 $\rightarrow$ 2 + 1). In this region of ρ_0, the periodic motion appears from the initial chaotic one. The examples of behaviour of spectra are presented in Figure 8. The investigation of the trajectories with another value of $a_i^{(0)}/R_i^{(0)}$ = 0.008 shows that the main rules of the distribution of regions with order and chaos are kept. The difference between this case and the one described above consists in the small displacement of borders for each region of ρ_0.

For initial systems which consist of four and five rings, the regular motion takes place only in cases when the initial set of rings disintegrates into two or three subsystems. These initial sets are more sensitive to chaotic interactions. In general, a regular situation when all four or five rings move together was not found. In the case N = 4, the regular motion occurs when the initial set of rings decays into two independent subsystems with either three and one rings (4 $\rightarrow$ 3 + 1), or two and two rings (4 $\rightarrow$2 + 2). In the case N = 5, the regular motion occurs accordingly to situations 5 $\rightarrow$ 2 + 2 + 1 and 5 $\rightarrow$ 3 + 2. Moreover, in both

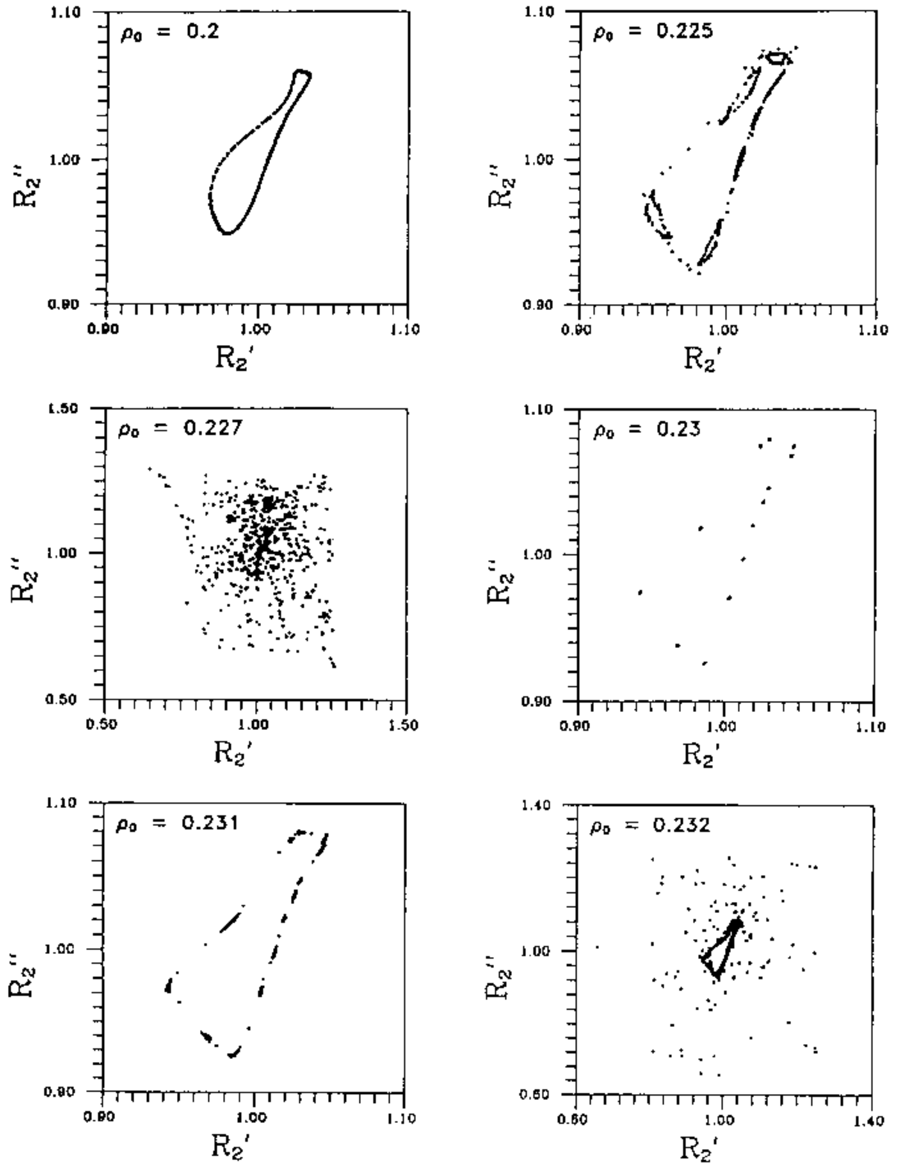


Figure 6.
Poincaré sections for three vortex rings

these cases there are the regions of values ρ_0, when the final set of rings moves chaotically by $4 \rightarrow 3 + 1$, or $5 \rightarrow 3 + 2$. In these situations one subsystem, which consists of three vortex rings, moves chaotic and another one – periodic. The detailed classification of types of the motion by initial conditions (8) for 3, 4 and 5 vortex rings has been made in[30].

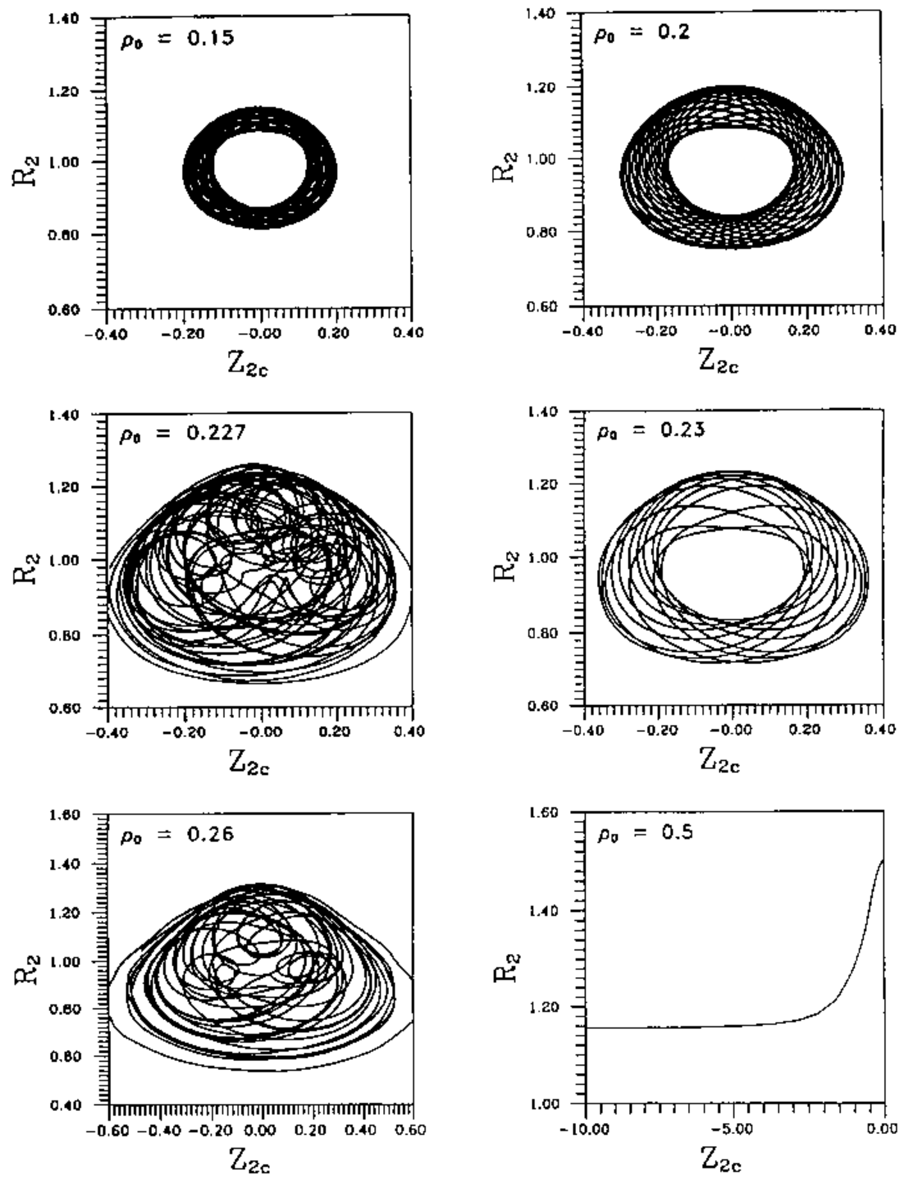


Figure 7.
Relative phase trajectories for three vortex rings

Mixing of passive fluid particles during vortex rings interaction
In different natural vortex phenomena or in experiments as a rule we observe the motion of passive fluid particles in the vorticity field. Only after special analysis can we determine the concrete parameters of vortices, namely their radii, values of the cross-sections and circulations. The problem of mixing of passive particles has a number of promising attractive applications.

The behaviour of scalars in the vector velocity field has been examined for many years. Common questions relating to this problem are

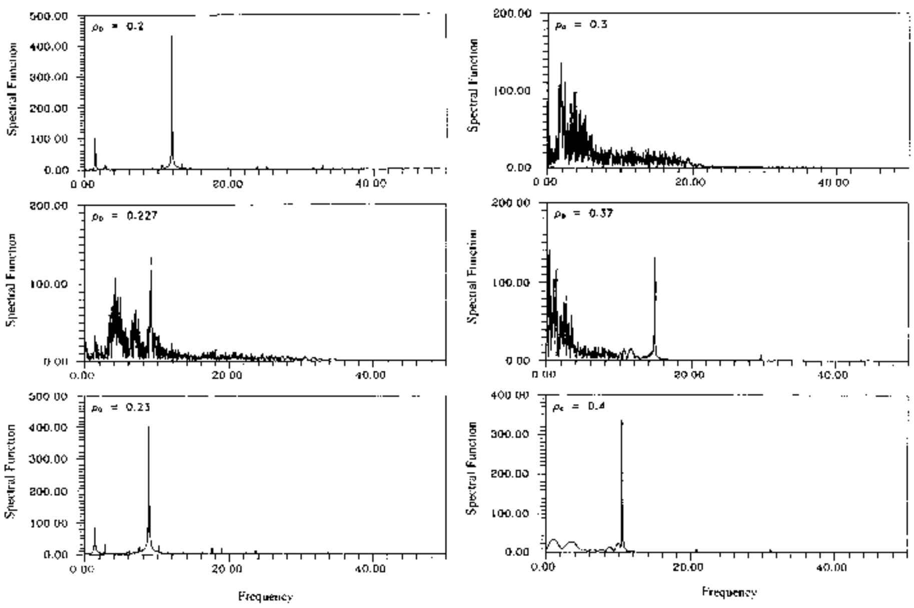


Figure 8. Power spectra for three vortex rings

presented in the works of Aref[32] and Ottino[33,34]. Some special cases of mixing in the field of vorticity were examined by Wiggins[35], Rom-Kedar *et al.*[36], Aref[37] and Zawadzki and Aref[38]. The passive particles can be considered as vortices with zero circulation. Such a method was applied to the 2D problem of behaviour of passive particles by interaction of point vortices in a paper by Meleshko *et al.*[39]. For the present case of coaxial vortex rings, it is suitable to present the passive particles as rings of zero circulation, which have a common axis with dynamic rings. The additional equations for passive particles with co-ordinates (Z,R) are:

$$\dot{Z} = \sum_{j=1}^{N} \frac{\kappa_j}{2\pi R} \frac{\partial I_j}{\partial R},$$

$$\dot{R} = -\sum_{j=1}^{N} \frac{\kappa_j}{2\pi R} \frac{\partial I_j}{\partial Z}, \qquad (10)$$

where

$$I_j = \sqrt{RR_j}\left[\left(\frac{2}{k_j} - k_j\right) K(k_j) - \frac{2}{k_j} E(k_j)\right],$$

$$k_j^2 = \frac{4RR_j}{(Z - Z_j)^2 + (R + R_j)^2}$$

with initial conditions $t = 0$: $R = R^{(0)}$, $Z = Z^{(0)}$. These equations must be solved together with system (3). As one can see, each passive particle moves in the velocity field generated by N vortex rings. These particles do not influence the rings. Additionally, the passive particles do not interact with one another.

As is well known, the problem of interaction of two vortex rings is integrable. The problem of mixing of passive particles in a field of vortex rings is integrable only by the motion of one vortex ring. In all other cases, the situation becomes non-integrable and must be analysed numerically. In this section, some examples of passive particles motion are given for the investigated cases of the interaction of two and three vortex rings. The fluid flow induced by a thin vortex ring is shown by fixed vortex co-ordinates in Figure 9. The closed streamlines create the so-called atmosphere of vortex rings. By the interaction of some rings the motion of the rings is not stable, and the rings also do not have their own isolated atmosphere. However, it is interesting to examine the process of mixing, when passive particles fill in the atmosphere of the isolated ring at the initial moment of time. Such a situation resembles the experiments by the generation of vortex rings when a tinted liquid is used. At the initial moment of time, up to 4,000 particles were distributed on the different closed streamlines. During the mixing process a situation occurs in which it becomes impossible to follow the topology of each streamline because of the large distance between two initially neighbouring points. In such cases, up to an additional 4,000 particles were distributed between those two initial points. The distance between these particles can be $\Delta Z = 0.0001$. These situations show the difficulties of the experiments, when associated with a loss of initial atmosphere or dilution. Both viscosity and dilution of the atmosphere are important reasons preventing the observation of the leapfrog motion during experimentation. Only in very precise experiments (for example [10] and [12]) has this phenomenon been perfectly observed.

The case of leapfrog motion is shown in Figure 10. The rings have the same radii and circulations $R_1^{(0)} = R_2^{(0)}$, $\kappa_1 = \kappa_2$, the same ratios $a_i^{(0)}/R_i^{(0)} = 0.01$, and they are in distance $Z_{12} = 1$ at the initial moment of time. The particles are marked into two colours for clarity in determining the atmosphere of the ring to which they belonged at the initial moment of time. The process of mixing is rather difficult. It is especially difficult to identify the particles in the

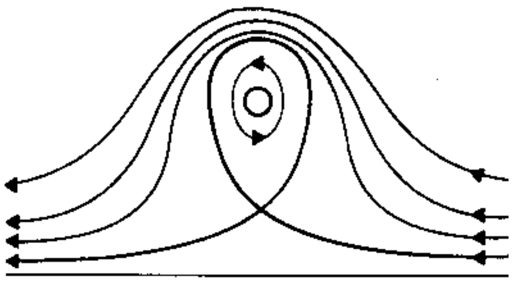


Figure 9.
Relative streamlines and atmosphere for vortex ring

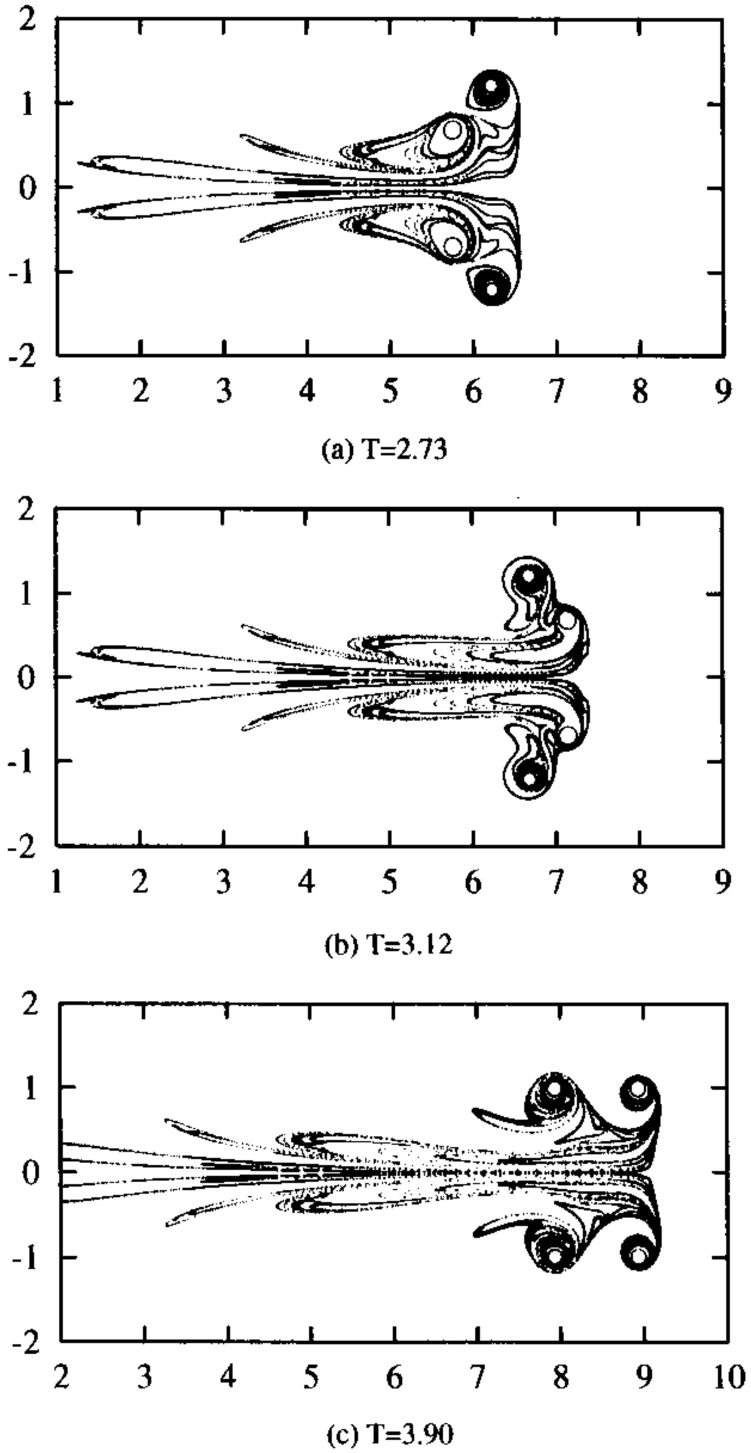


Figure 10.
Mixing passive particles by leapfrog motion of two vortex rings. Horizontal and vertical co-ordinates are axial (*Z*) and radial (*R*) positions of the particles respectively

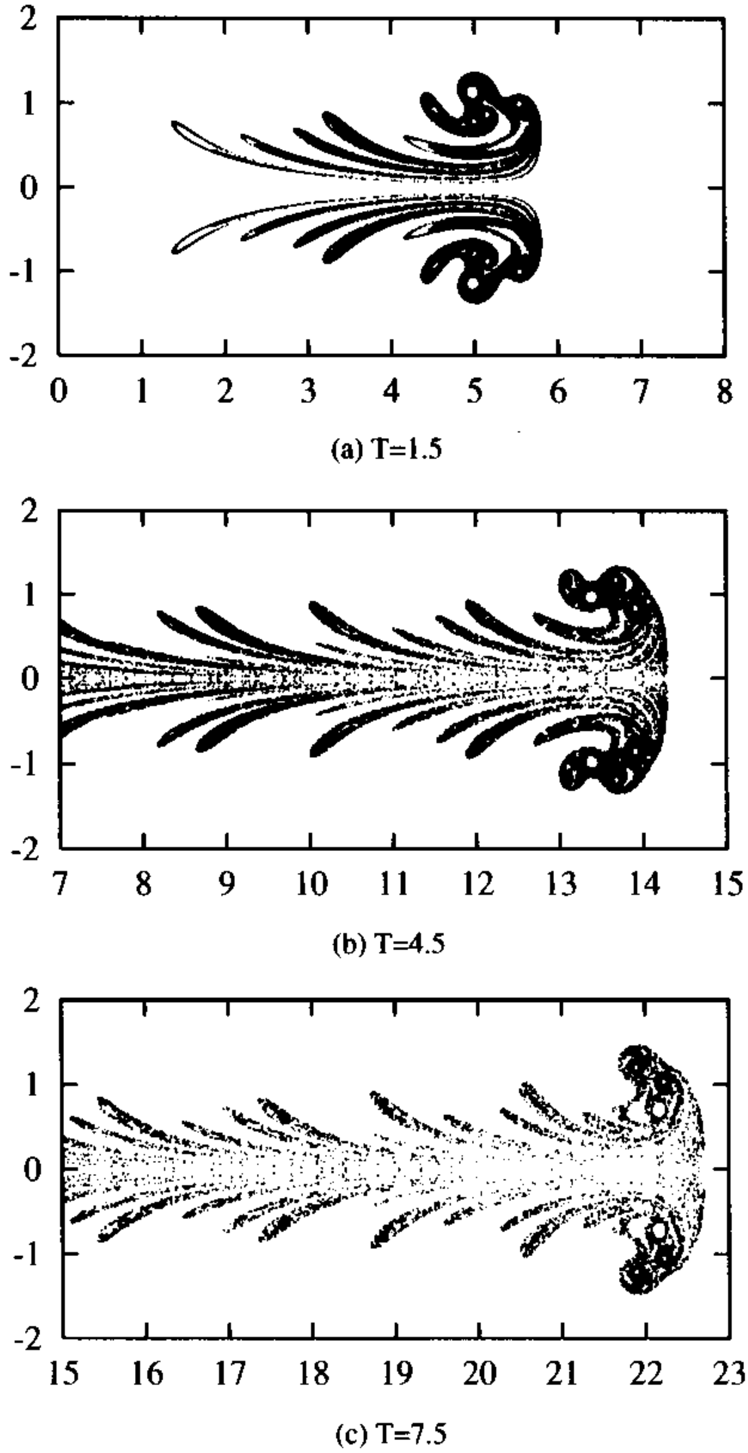
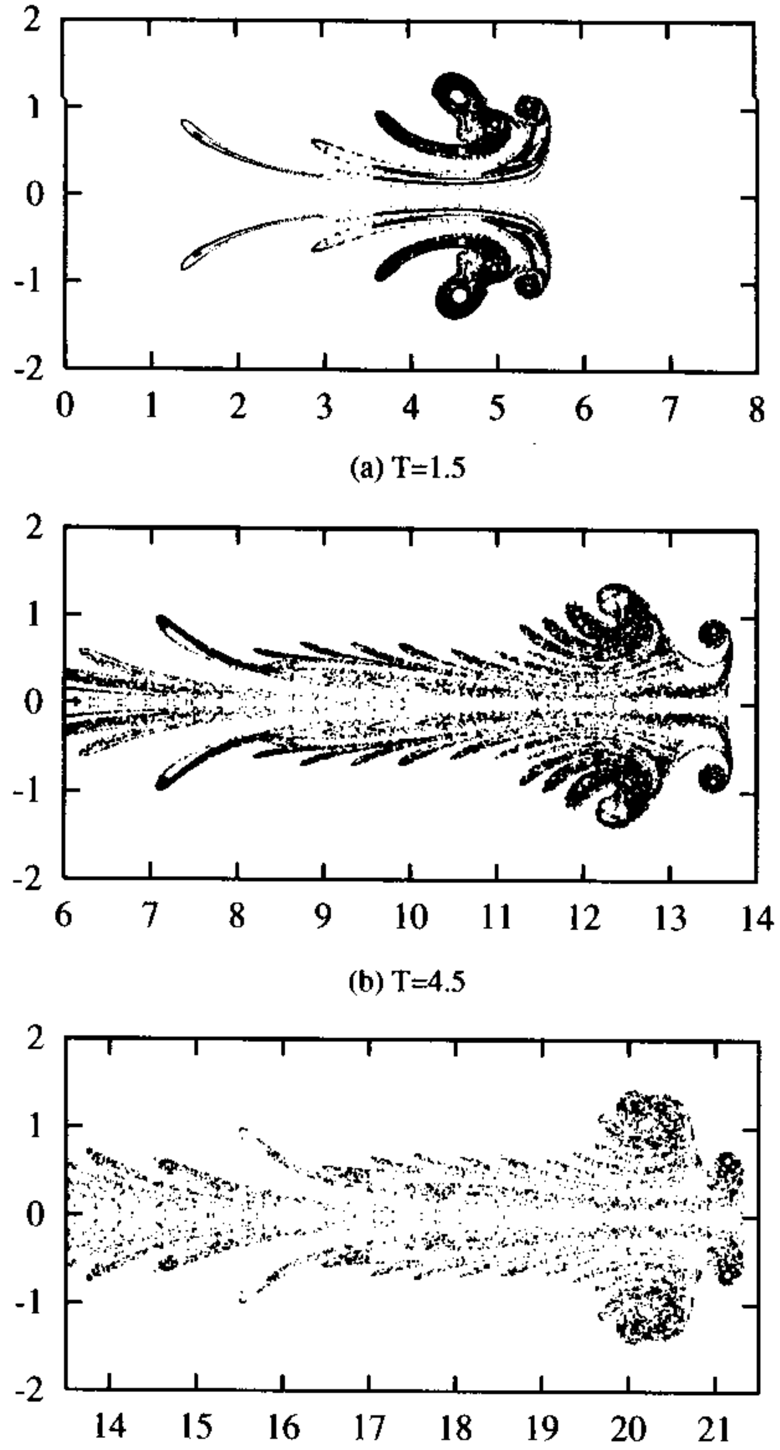


Figure 11.
Mixing passive particles by regular interaction of three vortex rings. Horizontal and vertical co-ordinates are axial (*Z*) and radial (*R*) posiitons of the particles respectively



(a) T=1.5

(b) T=4.5

Figure 12. Mixing passive particles by chaotic interaction of three vortex rings. Horizontal and vertical co-ordinates are axial (Z) and radial (R) positions of the particles respectively

experiments during mixing. However, there have been some successes in observing the leapfrog motion.

The behaviour of particles will be further complicated by the interaction of $N > 2$ rings. Two cases of motion of three vortex rings have been chosen. The first case corresponds to perfect order ($\rho_0 = 0.23$), and the other one to the chaos ($\rho_0 = 0.3$). The main question of this investigation was to distinguish a chaotic motion and regular one visually only, without additional physical analysis. The qualitative reflection of this process at the initial domain of interaction is shown in Figures 11 and 12. The case of mixing of particles by the interaction of three rings for $\rho_0 = 0.23$ is shown in Figure 11. It is possible to observe regular structures, especially in the wake. The case of chaos ($\rho_0 = 0.3$) in Figure 12 is shown. The difference in the behaviour of particles in these cases is quite clear.

Final remarks

In the dependence of a number of coaxial vortex rings and their motion can be described either by an integrable or non-integrable system of equations. In the non-integrable case, the system becomes very sensitive to chaos. Therefore, a high accuracy of numerical calculations is required. In spite of non-integrability, there are the quasi-periodic cases of three and more vortex rings. More stable configurations occur for three vortex rings. The cases when four or five vortex rings move quasi-periodically together were not found. The regular motion for such initial sets occurs only when they decay to several subsystems. The numerical visualization of flow by passive particles shows the opportunity to identify, qualitatively, the types of interaction. The question of the possibility of the existence of chaotic mixing domains of passive particles by quasi-periodic interaction of vortex structures is open and requires additional investigation.

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